Introduction to Spacetime diagrams in Special Relativity

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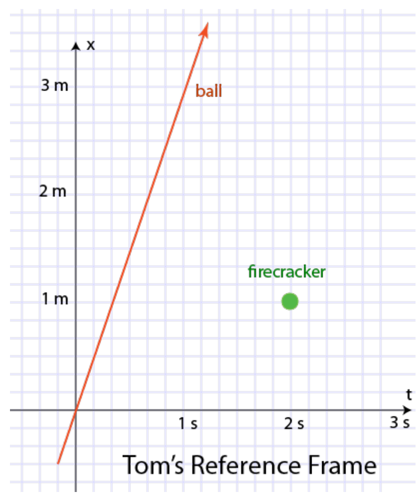
In his original paper in 1905, Einstein used several thought ("gedanken") experiments to illustrate the implications of the constancy of the speed of light. Other physicists found that it was easier to *visualize* the concepts of relativity and developed a variety of graphing techniques. We will use a simple graphing technique called a *Minkowski spacetime diagram*, or simply a "spacetime diagram," to illustrate and understand complex scenarios in special relativity. This diagram was originally developed by Hermann Minkowski in 1908 and is useful for objects that move at a substantial fraction of the speed of light.

Background. We are interested in the dynamics of a system of objects: where objects are and where they are going at any given time (i.e., position and velocity). Often the best way to visualize the motion of the objects is by making a plot of distance vs time. Consider the following two scenarios:

i. Tom throws a red ball with a horizontal (x) component of velocity of 3 m/s at the same time as he starts a stopwatch.
ii. Tom notices[1] a green firecracker explode at $x = 1$ m and at $t = 2$ s after he starts his stopwatch.

These two scenarios can be described on a plot of x vs t (right). The coordinate system can be described as "Tom's rest, or *reference*, frame." Tom considers himself to be at rest in his own frame, but nothing is otherwise known about how Tom may be moving with respect to another frame.



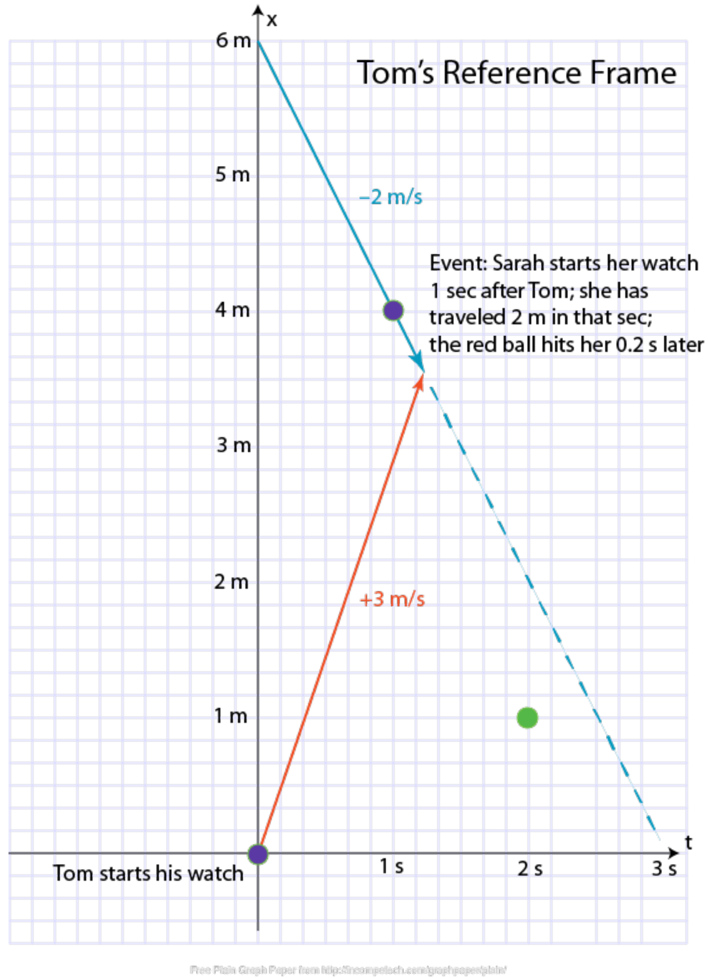
Conceptual Questions to Ponder
(see answers at the end):

- What is the physical significance of the origin in the plot?
- How would *Tom's* location at various times on the plot be represented?

[1] The meaning of the word "notice" seems unambiguous, but as we will see, this is true only in the "classical" viewpoint. When we begin speaking about special relativity later in the document, we will distinguish the difference between the time a person sees (notices) the results of an event, and the location and time of the event itself.

Let's now assume Tom sees his good friend Sarah cruising down the road at 2 m/s towards him She is moving along his x-axis and the red ball he threw is on a collision course with her. As it happens, Sarah is 6 m away from Tom when he throws the ball, and it takes her one second to react after he has thrown the red ball, at which time she starts her own stopwatch. What does the above distance vs time plot look like from Sarah's frame of reference?

Before we answer this question, it will help to superimpose Sarah's motion onto Tom's plot:



From Tom's perspective, Sarah (blue) is moving at –2 m/s as the red ball he threw moves in her direction at +3 m/s. From simple constant-acceleration kinematics, we find that the red ball meets Sarah at (t = 1.2 s, x = 3.6 m) in Tom's reference frame. Sarah continues to approach and reaches Tom's location at t = 3 s.

In contrast, Sarah considers herself to be at rest. She sees Tom moving in her direction at 2 m/s and the red ball moving in her direction at 2+3=5 m/s.

To prove this result mathematically, we note a connection to the relative motion vector equation:

$$\vec{v}_{13} = \vec{v}_{12} + \vec{v}_{23}$$

It is traditional to distinguish the two reference frames by labeling one with primes; Tom's reference frame is referred to as S, and Sarah's as S'. Assign the red ball to subscript 1, and subscripts 2 and 3 to the respective reference frames. Then,

$$\vec{v}_{1S} = \vec{v}_{1S'} + \vec{v}_{S'S} \quad \rightarrow \quad +3 \text{ m/s} = \vec{v}_{1S'} + (-2 \text{ m/s}) \quad \rightarrow \quad \vec{v}_{1S'} = +5 \text{ m/s}$$

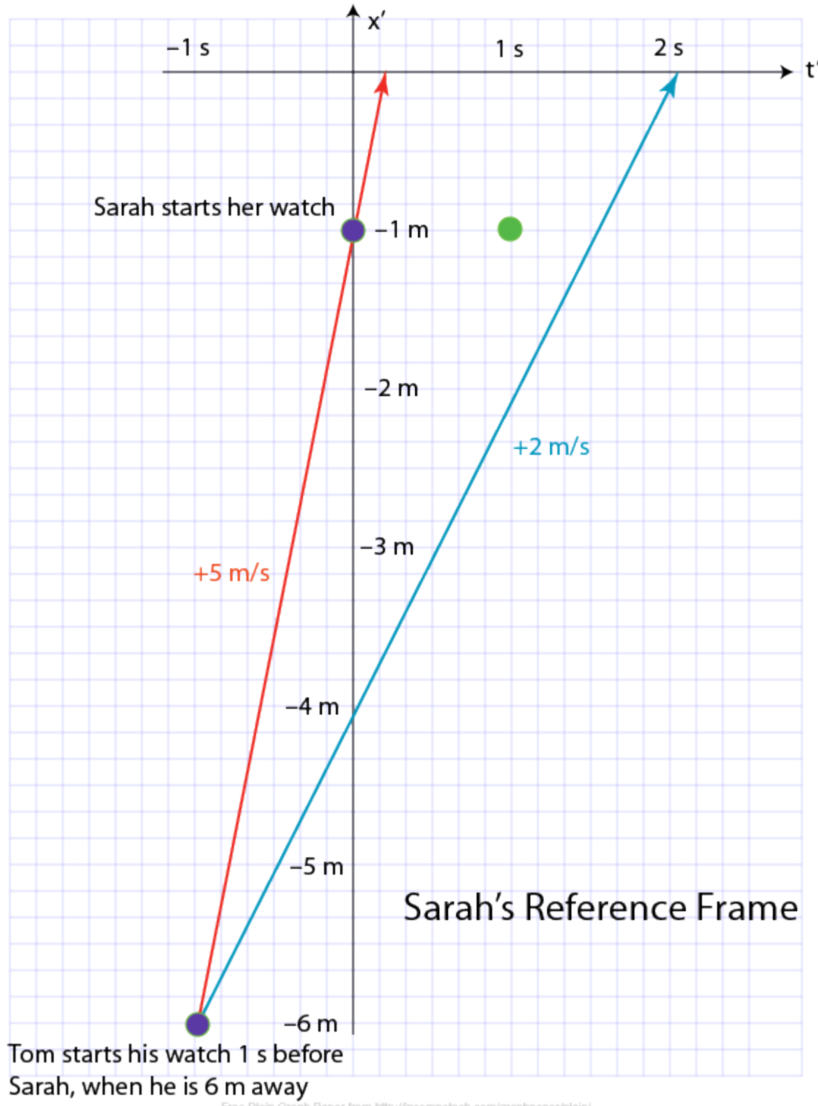
We can now think about drawing Sarah's plot, but we need more information to lock it down: Tom's "time zero" is one second before Sarah's "time zero." This means that the numbers on Sarah's stopwatch *lag* the same numbers on Tom's stopwatch. To properly locate this relationship on Sarah's plot, we note that the red ball is 1 m away from her when she starts her stopwatch. Using the same constant-acceleration kinematics as in Tom's case, we find that the red ball meets Sarah at (t' = 0.2 s, x' = 0 m) in Sarah's coordinate system.

To complete the picture, Sarah notices the firecracker at t' = 1 s, as opposed to Tom's t = 2 s (because she started her stopwatch later).

The firecracker is 1 m in front of Tom and thus 5 m in front of Sarah when he throws the red ball, but it doesn't explode until 2 s after. During this time, Sarah has moved 4 m in Tom's direction. The firecracker is therefore at her 1 m mark according to Sarah.

The final result from Sarah's frame of reference is shown to the right. In this case, the blue trajectory is Tom's motion in Sarah's frame.

In either reference frame, the 2 m/s value describes the relative speed of the reference frames ($v_{s's}$). In this example, this relative speed is a "closing" speed. Notice that the same red ball appears to have *different* speeds in each reference frame, because Tom and Sarah are moving with respect to each other.



Sarah's plot *depicts exactly the same occurrences as Tom's plot*. These occurrences are what we will soon call "events." The conversion of the description from one reference frame (Tom's) to another (Sarah's) is known as a "Galilean transformation." The following features characterize this transformation:

- The frames of reference are moving with respect to each other at a constant velocity.
- The position coordinate (x) of an event in one frame will typically have a different position coordinate (x') in another frame.
- The time coordinate (t) of an event in one frame may have a different time coordinate (t') in another frame, but the ***passage*** of time will be the same in both frames ($\Delta t = \Delta t'$).
- Objects with a velocity in one frame will have a different velocity in another frame.

The Galilean Transformation comprised our understanding of relative motion and how measurements compare between frames up until 1905, when Einstein first published his paper on special relativity. In most applications, Galilean transformations remain correct to a very high accuracy; however, one fundamental observational fact, that the speed of light is measured to be the same in all reference frames, forces us to modify the Galilean ideas and gives rise to some very interesting and non-intuitive results when two reference frames are moving relative to one another at speeds significant compared to the speed of light.

To begin our own "transformation" to this new way of thinking, we first introduce some important terminology, and then modify our use of these Galilean kinematics plots to our advantage.

Terminology. In the jargon of spacetime diagrams, the green point on Tom's and Sarah's plots is an *event* and the red or blue trajectories are *worldlines*.

An **event** is anything that can be characterized by a single point on a spacetime diagram (or on a position vs time graph). An event *must* have *both* a time and a place, and specifically refers to where and when the event ***actually*** happened within that reference frame. We distinguish this from when an event is seen or noticed – seeing or noticing an event is actually a *second* event with its own time and place. Examples of events include the point in time and space at which Tom threw the red ball, as well as the point and time when the ball collides with Sarah. Yet another event is when and where the firecracker exploded, and still another event is when and where Tom sees the light from the firecracker explosion. All of the events are known to occur by both Tom and Sarah, but each of them will consider the events to have different time and position coordinates in their respective frames.

A **worldline** can be defined as 'a curve in spacetime joining the positions of a particle throughout its existence.' Each object has its own worldline within a given reference frame. The worldline thus describes the entire history and future of the object. A worldline is composed of an infinite continuum of events (e.g., the location of the red ball at one time, the location of the red ball an infinitesimal time later, *ad infinitum*) and describes the motion of an object within that reference frame. Objects moving at a constant speed have straight worldlines. Objects that accelerate have curved worldlines; however, we typically won't consider accelerations in our treatment of special relativity.

Building a Spacetime Diagram. Most of the problems we will consider in special relativity only have one relevant position coordinate, which is the direction of relative motion between objects we wish to study. We call this coordinate x and plot it on the *horizontal* axis. We plot time on the *vertical* axis. This is reversed compared to what you are used to, but as we work through some examples, you will see that this is sensible for relativistic scenarios. In addition to this reversal, there is one other important difference: time is rescaled by the speed of light; that is, ct (rather than t) is plotted on the vertical axis. As a consequence, the unit of time is replaced by its meter equivalent, so that both axes are measured in the same units. As a result, the time "1 s" is replaced by "the distance light travels in 1 s."

To obtain a worldline in the new system of units, start with the known equation of motion and then manipulate it in a way that is useful for our purposes, but does not change the content of the equation. To wit:

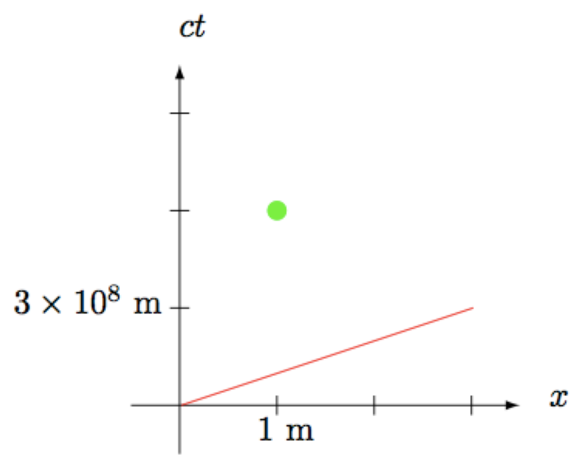
$$x = vt \quad \rightarrow \quad x = \frac{v}{c}(ct)$$

Because we want *ct* on the y-axis, the velocity term is moved to the opposite side to obtain

$$ct = \frac{c}{v}x$$

This equation tells us that **the slope of the worldline of an object traveling at velocity *v* relative to a frame S is *c*/*v***.

The red ball on a spacetime diagram in Tom's reference frame then looks like the figure to the right. In this example, the red ball is moving very slowly compared to the speed of light. Thus the two axes are most naturally presented with scales that differ by eight orders of magnitude; however, in problems that are interesting from the perspective of special relativity, the scale of the two axes (*x* and *ct*) will be of comparable size.



Here are some key concepts required to draw correct spacetime diagrams:

- A spacetime diagram applies to a specific reference frame. You can draw a spacetime diagram for any reference frame, but the diagram will not look the same when drawn in a different reference frame.
- Events on any horizontal line have the same value for *t* and are *simultaneous* in the reference frame.
- Events on any vertical line have the same value for *x* and thus occur at the same position.
- The spacetime origin ($x = 0$, $t = 0$) is the present time and location of the observer centered in this reference frame. Events with $t > 0$ are in the *future* and events with $t < 0$ are in the *past* of this observer.
- The *location* of the origin can be chosen to make the solution of a problem as convenient as possible, just as for regular kinematics.

We present below two examples of drawing spacetime diagrams to help clarify. Your first Relativity studio will refer to these examples, so it is important that you read them.

Scooby and Shaggy

Scooby and Shaggy, stationed at a remote astrophysical observatory, are tasked with monitoring the life cycle of nearby stars. Scooby looks out the window just in time to *see* a distant star go nova. He checks his charts and learns that that particular star is 1.8 billion km away. Shaggy immediately jumps in his shaggyship and jets away at his maximum speed of 0.5*c* towards the star. Thirty minutes later Scooby observes yet another star go nova; this one is 2.7 billion km away in the opposite direction. Scooby immediately signals Shaggy on the radio to let him know about the second nova. When he receives the transmission, Shaggy decides that the latter explosion is of greater scientific interest so he slams on the brakes and reverses direction towards the remnants of the second star. Our goal is to describe these events with a spacetime diagram.

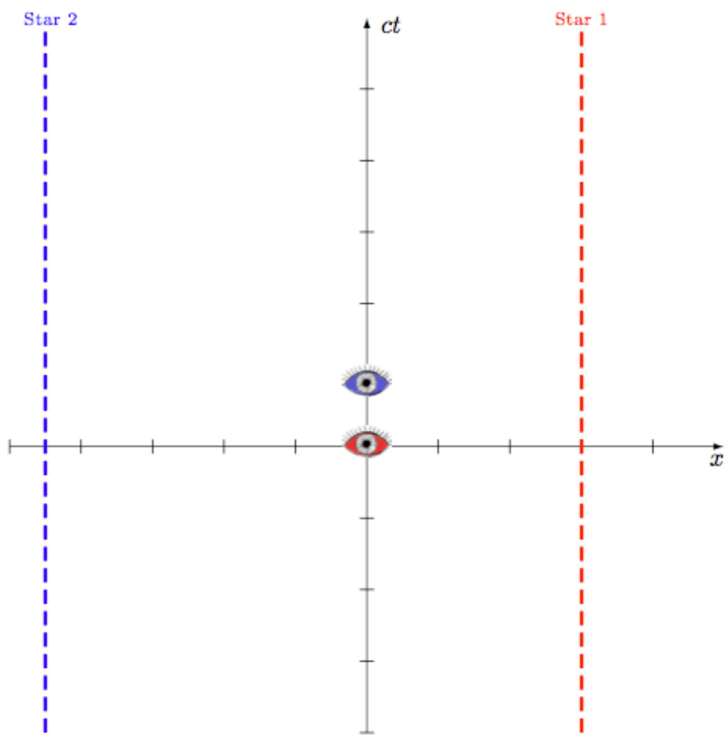
Because it initially seems like the simplest approach, we select Scooby's reference frame for the spacetime diagram. We choose to define $t = 0$ as the time when Scooby *sees* the first star explode (in general, note the extreme importance of distinguishing between when an event *occurs* and when it is *observed*). The events and worldlines provided explicitly in the example are shown below in Scooby's frame of reference:

The locations of the stars are known independently from Scooby's star charts. For reference, the tick marks are 6×10^{11} meters (0.6 billion km) apart. Their worldlines are shown in dotted form and are considered to be at rest with respect to Scooby's observatory.

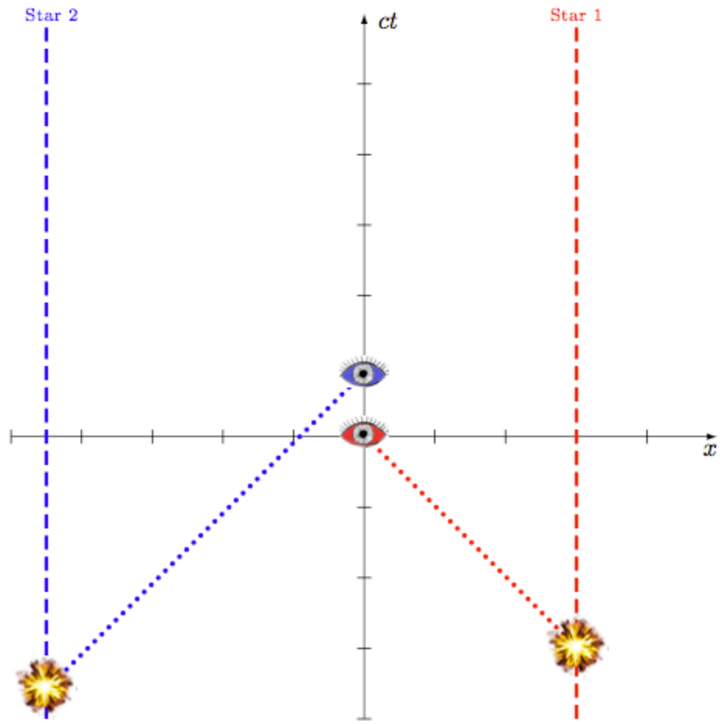
The red eye is the event "Scooby *sees* the 1st star explode."

The blue eye is the event "Scooby *sees* the 2nd star explode."

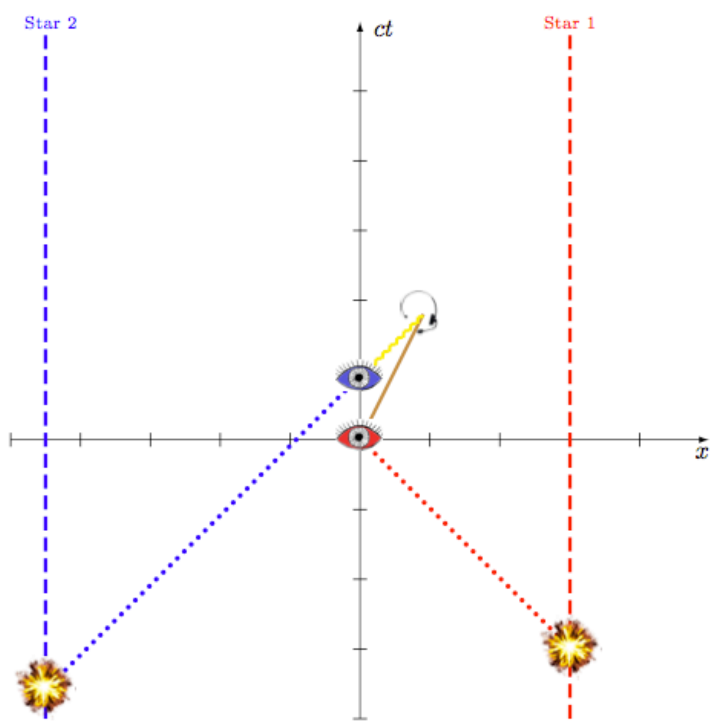
Conceptual Question to Ponder: Does it make sense that the second event is approximately 1 tick mark (6×10^{11} meters) along the *ct*-axis from the first event? (see answer at the end).



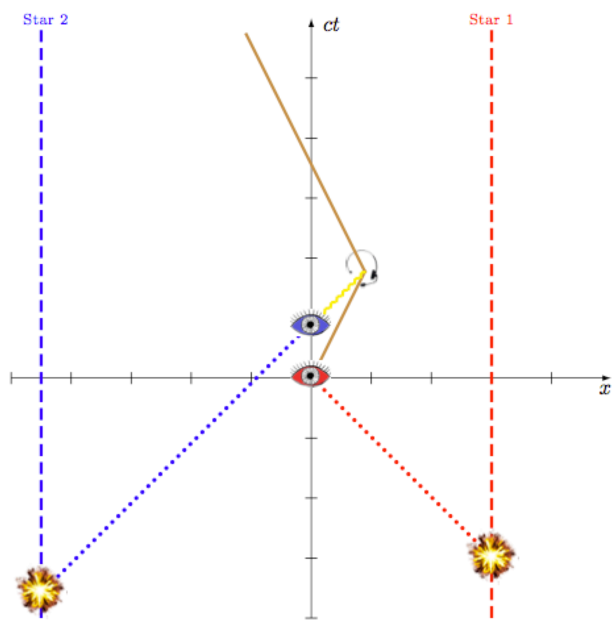
From the above starting diagram we can trace the world lines of the light given off from the novae explosions to figure out exactly *when* the stars self-destructed. The worldlines of light (since light moves at speed c) always have a slope of ±1. These are superimposed on the diagram. The diagram immediately informs us that, in Scooby's frame of reference, Star 2 exploded first, and Star 1 second, even though Scooby *sees* the events in the reverse order. The diagram immediately shows why this is the case: light from star 1 has to travel a smaller distance.



The next step is to add Shaggy and the radio communication to the spacetime diagram. Shaggy's initial speed is $+0.5c$, which has a slope of +2 on the diagram (brown worldline). The radio signal (slope = +1, yellow) on its way to Shaggy's headset is not sent until Scooby *sees* the second nova:



When Shaggy gets the signal to turn around, we add in the worldline for the last leg of his journey. Shaggy has reversed direction, so his new slope is –2. As a result of the reversal, he eventually returns to Scooby's location at a later time, and continues to the second nova.



Given the size of the tick marks on the diagram, it is possible to determine all the numerical values by reading them off the diagram. At times, we will use the equations of relativistic kinematics to compare the results to those on the diagrams.

<u>Answers to Conceptual Questions from page 1:</u>

- What is the physical significance of the origin in the plot?

  The origin is Tom's position at t = 0, which we've defined as when he starts his stopwatch. Note that you may make a plot where time is shifted that still represents Tom's reference frame; in this problem we drew the ball toss starting at t = 0 for our convenience.

- How would *Tom's* location at various times on the plot be represented?

  Since Tom is not moving within his reference frame, he remains at x = 0 at all times. Thus, his location is represented by the horizontal (time) axis.

<u>Answer to Conceptual Question from page 6:</u>

- Does it make sense that the second event is approximately 1 tick mark (6×10^{11} meters) along the *ct*-axis from the first event?

  We know that the two events are 30 minutes apart in Scooby's reference frame. If we calculate the distance light travels in 30 minutes:

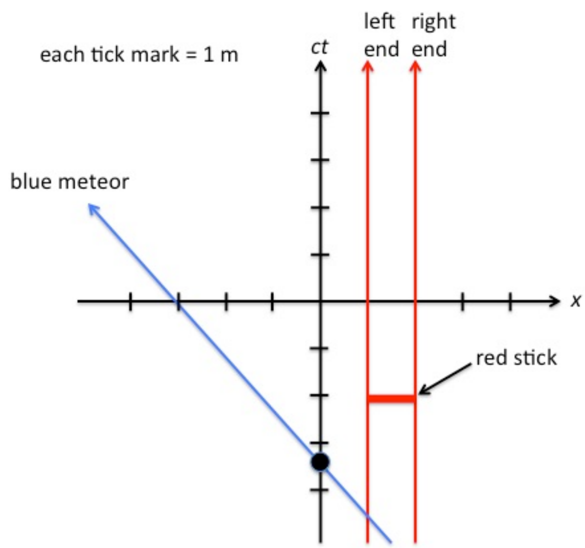
$$ct = (3\times10^{8}\text{ m/s})(60\text{ s/min})(30\text{ min}) = 5.4\times10^{11}\text{ m/s}$$

It's close – the second event is slightly below the 1st tick mark.

You Don't Know Jack
(will be covered in studio)

Jack is sitting at a table on a spaceship that passes by a lonely blue space meteor. The relative speed between Jack's spaceship and the meteor is 0.9c. Jack observes the meteor moving in the direction of his negative x-axis, while an observer at rest on the meteor would observe the spaceship moving along their positive x-axis. On board the ship with Jack is a red meter stick – it rests on the table with its ends at $x = +1$ m and +2 m. The meter stick is at rest in Jack's reference frame; therefore, its red world line has a slope of $c/0$ (infinity), which describes a vertical line. The blue space meteor, however, is moving past Jack with a velocity of –0.9c, so its blue world line has a slope of c/(–0.9c) = –1.11. Notice the advantage gained by plotting time "as a distance": all slopes on a spacetime diagram are dimensionless (they are merely positive or negative numbers).

The spacetime diagram in Jack's reference frame (right) shows the worldlines of the red meter stick and the blue space meteor, as just described. Jack's location is along the ct axis, at $x = 0$. Although Jack is at rest in the conventional sense, he is in fact traversing through spacetime: *he is moving into his future*.



The black dot represents the spacetime coordinates of the *event* we denote as "the time when and the place where the blue space meteor passes Jack."

Imagine now that Jack's captain, sitting in the ship to Jack's left, slides a green meter stick towards Jack on the table at a velocity of 0.8c with respect to Jack, oriented such that it is parallel to the red stick. Jack watches the green stick slide by the red stick (very quickly!). The green stick is moving past Jack with a velocity of 0.8c, so its green worldline has a slope of c/(0.8c) = +1.25. Jack starts his stopwatch ($ct = 0$) at the moment that the left end of each of the meter stick lines up. These occurrences are shown in the figure below right.

Note an interesting omission from the figure – what happened to the right end of the green meter stick? It's missing from the figure, because something peculiar happens which cannot be explained yet: according to Jack, the right ends of each meter stick won't line up at the same time the left ends do, and the two meter sticks will appear to be different lengths. This "relativity of simultaneity," and its consequence, "length contraction," will be addressed in the 2nd and 3rd Relativity modules.